

# On tensor products of positive representations of split real quantum Borel subalgebra $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$

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## Abstract

We studied the positive representations  $\mathcal{P}_{\lambda}$  of split real quantum groups  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to the Borel subalgebra  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ . We proved that the restriction is independent of the parameter  $\lambda$ . Furthermore, we prove that it can be constructed from the GNS-representation of the multiplier Hopf algebra  $\mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$  constructed earlier, which allows us to decompose their tensor product using the theory of multiplicative unitary. In particular, the quantum mutation operator can be constructed from the multiplicity module, which will be an essential ingredient in the construction of quantum higher Teichmüller theory from the perspective of representation theory, generalizing earlier work by Frenkel-Kim.

**Keywords.** Positive representations, split real quantum groups, modular double, GNS-representation, higher Teichmüller theory, quantum dilogarithm

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## 1 Introduction

In this paper we studied the positive representations  $\mathcal{P}_{\lambda}$  of split real quantum groups  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to its Borel part, show that the representation is independent of the parameter  $\lambda$ , and it is closed under taking tensor product. In particular, this representation-theoretic approach generalized the construction given by Frenkel-Kim [11] for the quantum plane  $\mathcal{B}_{q\tilde{q}}$ , providing a new construction of the quantum mutation operator  $\mathbf{T}$  [11, 20] necessary to construct a version of quantum higher Teichmüller theory [8, 9]. Moreover, the use of  $C^*$ -algebra and the theory of multiplicative unitary on  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  generalized the harmonic analysis of the quantum plane studied in [13]. Furthermore, the unitary transformations by the remarkable quantum dilogarithm functions also provide rich combinatorial insight into the tensor category structure of the positive representations, and giving hints to a conjecture generalizing the Stone-von Neumann's Theorem on higher rank quantum Borel algebra.

The notion of the *positive principal series representations*, or simply *positive representations*, was introduced in [10] as a new research program devoted to the representation theory of split real quantum groups  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ . It uses the concept of modular double for quantum groups [4, 5], and has been studied for  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$  by

Teschner *et al.* [1, 25, 26]. Explicit construction of the positive representations  $\mathcal{P}_\lambda$  of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  associated to a simple Lie algebra  $\mathfrak{g}$  has been obtained for the simply-laced case in [14] and non-simply-laced case in [15], where the generators of the quantum groups are realized by positive essentially self-adjoint operators. Furthermore, since the generators are represented by positive operators, we can take real powers by means of functional calculus, and we obtained the so-called *transcendental relations* of the (rescaled) generators:

$$\tilde{\mathbf{e}}_i = \mathbf{e}_i^{\frac{1}{b_i^2}}, \quad \tilde{\mathbf{f}}_i = \mathbf{f}_i^{\frac{1}{b_i^2}}, \quad \widetilde{K_i} = K_i^{\frac{1}{b_i^2}} \quad (1.1)$$

giving the self-duality between different parts of the modular double, while in the non-simply-laced case, new explicit analytic relations between the quantum group and its Langlands dual have been observed [15].

In the case of the modular double  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ , the positive representations is shown to be closely related to the space of conformal blocks of Liouville theory, and there are direct relations between the two and the quantum Teichmüller theory, related by the fusion and braiding operations [24, 30]. In particular, it is shown in [26] that for  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ , the family of positive representations  $\mathcal{P}_\lambda$  is closed under taking tensor product, where the Plancherel measure  $\mu(\lambda)$  is given by the quantum dilogarithms:

$$\mathcal{P}_{\lambda_1} \otimes \mathcal{P}_{\lambda_2} \simeq \int_{\mathbb{R}_+} \mathcal{P}_\lambda d\mu(\lambda). \quad (1.2)$$

The fusion relations of Liouville theory are then provided precisely by the decomposition of the triple tensor products. Together with the existence of a universal  $R$ -operator (constructed for general  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  in [16]) giving the braiding structure, the braided tensor category structure may give rise to a new class of topological quantum field theory (TQFT) in the sense of Reshetikhin-Turaev [27, 28, 34]. Therefore one of the major remaining unsolved problem in the theory of positive representations for higher rank is the structure under taking tensor products, and it is expected that the decomposition will be related to the corresponding fusion relations of more general non-compact CFT's such as the Toda conformal field theory [12, 35].

On the other hand, the quantum Teichmüller space was constructed in [11] from the perspective of representation theory of certain Hopf algebra, namely, the modular double of the quantum plane, which is just the Borel part of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ . More precisely, (the modular double of) the quantum plane  $\mathcal{B}_{q\tilde{q}}$  is a non-compact version of the quantum torus generated by two sets of commuting generators  $\{E, K\}$  and  $\{\tilde{E}, \tilde{K}\}$  such that

$$KE = q^2 EK, \quad \tilde{K}\tilde{E} = \tilde{q}^2 \tilde{E}\tilde{K}. \quad (1.3)$$

The quantum plane has a canonical representation on  $\mathcal{H} \simeq L^2(\mathbb{R})$  such that the generators  $E, K$  are represented by positive self-adjoint operators

$$E = e^{-2\pi b p}, \quad K = e^{-2\pi b x}, \quad \tilde{E} = e^{-2\pi b^{-1} p}, \quad \tilde{K} = e^{-2\pi b^{-1} x}, \quad (1.4)$$

where

$$q = e^{\pi i b^2}, \quad \tilde{q} = e^{\pi i b^{-2}}, \quad p = \frac{1}{2\pi i} \frac{\partial}{\partial x}, \quad (1.5)$$

and  $0 < b < 1, b \in \mathbb{R} \setminus \mathbb{Q}$ . Using this fact, Frenkel-Kim [11] showed that  $\mathcal{H}$  is closed under taking the tensor product and decomposes as

$$\mathcal{H} \otimes \mathcal{H} \simeq M \otimes \mathcal{H}, \quad (1.6)$$

where  $M \simeq \text{Hom}_{B_{q\tilde{q}}}(\mathcal{H}, \mathcal{H} \otimes \mathcal{H})$  is the multiplicity module with  $\mathcal{B}_{q\tilde{q}}$  acting trivially. Upon identification of  $\mathcal{H}$  and  $M$  with  $L^2(\mathbb{R})$ , this can be expressed by a transformation given by the quantum dilogarithm function (cf. (6.2)). Then the canonical isomorphism

$$(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \simeq \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3) \quad (1.7)$$

yields an operator on the multiplicity modules

$$\mathbf{T} : M_{43}^6 \otimes M_{12}^4 \simeq M_{23}^5 \otimes M_{15}^6, \quad (1.8)$$

which is often called the *quantum mutation operator*, and by construction satisfies the pentagon relation. Together with another operator  $\mathbf{A}$  where  $\mathbf{A}^3 = 1$  coming from the identification of the dual representations from the antipode  $S$  of  $\mathcal{B}_{q\tilde{q}}$ , one recovers Kashaev's projective representation of the mapping class groupoid  $\mathfrak{G}$  [20], and we can apply it to quantization of Teichmüller space for various surfaces as shown e.g. in [2, 7, 20]. The operators  $\mathbf{T}$  and  $\mathbf{A}$  correspond essentially to the fusion and braiding relations on the conformal field theory side.

Note that the quantum plane is just the Borel subalgebra of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ , and the canonical representation on  $\mathcal{H}$  satisfies the properties of a positive representation. Therefore in order to generalize to higher rank, a natural candidate is to consider the Borel subalgebra  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  of the modular double of more general split real quantum groups  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ . It follows that the main ingredient needed to construct a version of quantum higher Teichmüller theory [8, 9] is the construction of the quantum mutation operator  $\mathbf{T}$  as described above, coming from the decomposition of tensor products of positive representations of the Borel part.

In this paper, we proved the following theorem (Theorem 3.1, Corollary 5.6)

**Main Theorem.** *The positive representations  $\mathcal{P}_{\lambda}$  of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to the Borel part  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  is independent of the parameters  $\lambda$ . Moreover, the tensor products of  $\mathcal{P}_{\lambda} \simeq \mathcal{P}$  decomposes as*

$$\mathcal{P} \otimes \mathcal{P} \simeq M \otimes \mathcal{P}, \quad (1.9)$$

where  $M \simeq L^2(\mathbb{R}^N)$  is the multiplicity module with  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  acting trivially. Here  $N$  is the dimension of  $\mathfrak{b}_{\mathbb{R}}$ .

We give a new construction of positive representations of the Borel part by means of the notion of multiplier Hopf algebras introduced in [21, 32]. We used the  $C^*$ -algebraic version of the Borel part

$$\mathcal{A} := \mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$$

constructed in [16, 17], and apply the theory of Gelfand-Naimark-Segal (GNS) representations of  $C^*$ -algebras. This generalized the harmonic analysis of the quantum plane studied in [13], which was introduced to study harmonic analysis of the quantum space of functions  $L^2(SL_{q\tilde{q}}^+(2, \mathbb{R}))$ . A useful property of this construction is the existence of a unitary operator  $W$ , known as the multiplicative unitary, which gives the desired intertwiners

$$W\Delta(x)W^* = 1 \otimes x, \quad x \in \mathcal{A} \quad (1.10)$$

between the tensor products and its decomposition with a trivial multiplicity module, provided that all representations  $\mathcal{H}$  of  $\mathcal{A}$  are equivalent. Then another main result of this paper is the following (Theorem 5.5):

**Main Theorem.** *For any simple Lie algebra  $\mathfrak{g}$ , the GNS representation  $\mathcal{P}_{GNS}$  constructed by left multiplication on  $\mathcal{A} = \mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$  is unitary equivalent to the positive representation  $\mathcal{P}_{\lambda} \simeq \mathcal{P}$  of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to its Borel part.*

With the various identifications of the positive representations of the quantum Borel subalgebra  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ , it leads us to conjecture a generalization of the Stone-von Neumann's Theorem (Conjecture 3.2) to the higher rank using the notion of integrable representation of the quantum plane.

Finally, we provide concrete examples for type  $A_1, A_2$  and  $A_3$  to show how the decompositions look like. In particular, these equip us with explicit expressions for the quantum mutation operators  $\mathbf{T}$  needed in order to construct candidates for the quantum higher Teichmüller theory. Together with the (unitary) antipode  $R_S$  defined in [13] and the identification of the dual representations [11], this give us the desired construction of the projective representation of Kashaev's mapping class groupoid  $\mathfrak{G}$  which acts on a Hilbert space  $L^2(\mathbb{R}^N)$  with higher functional dimension. This is expected to be a new class of representations for  $\mathfrak{G}$ . Details will appear elsewhere in a joint work with Hyun Kyu Kim.

The paper is organized as follows. In Section 2 we recall several background theories needed in this paper. We fixed the notations of a general quantum group  $\mathcal{U}_q(\mathfrak{g})$ , and preset a summary of the positive representations and their properties. Then we recall the definitions and some properties of the quantum dilogarithm functions  $G_b(x)$  and its variant  $g_b(x)$ . Next we recall the Lusztig's isomorphism relating simple and non-simple roots in the positive setting as constructed in [16]. Finally, we recall the theory of GNS representation of  $C^*$ -algebra, and introduce the multiplicative unitary  $W$ , which is studied extensively in the context of locally compact quantum groups [21, 31].

In Section 3, we consider the restriction of the original positive representations  $\mathcal{P}_{\lambda}$  to the Borel subalgebra, and show that the representation is independent of the

parameters  $\lambda$ . In Section 4, we modify the definition of the  $C^*$ -algebraic version  $\mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$  of the Borel subalgebra given in [16], and consider the GNS-representation obtained from the action by left multiplication. As in the construction of positive representations, we consider the rank 2 case which gives us the unitary transformation needed for changing the reduced expression of the longest element  $w_0$  of the Weyl group by the Coxeter relations, and finally we introduce a twisting to bring the expression into a canonical form. In Section 5, we prove that the representation  $\mathcal{P}_{GNS}$  constructed from the GNS representation is unitary equivalent to the one constructed earlier in [14, 15], thus proving the Main Theorem via the theory of multiplicative unitary. Finally in Section 6, we give concrete constructions of the decomposition of tensor products for  $\mathfrak{g}_{\mathbb{R}}$  of type  $A_1, A_2$  and  $A_3$ , while for the simplest case  $A_1$  we also look at the transformation provided by the multiplicative unitary explicitly.

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## 2 Preliminaries

Throughout the paper, we will fix once and for all  $q = e^{\pi \mathbf{i} b^2}$  with  $\mathbf{i} = \sqrt{-1}$ ,  $0 < b^2 < 1$  and  $b^2 \in \mathbb{R} \setminus \mathbb{Q}$ . We also denote by  $Q = b + b^{-1}$ .

### 2.1 Definition of $\mathcal{U}_q(\mathfrak{g})$

In order to fix the convention we use throughout the paper, we recall the definition of the Drinfeld-Jimbo quantum group  $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$  where  $\mathfrak{g}$  is a simple Lie algebra of general type [3, 18]. Let  $I = \{1, 2, \dots, n\}$  denotes the set of nodes of the Dynkin diagram of  $\mathfrak{g}$  where  $n = \text{rank}(\mathfrak{g})$ .

**Definition 2.1.** Let  $(-, -)$  be the  $W$ -invariant inner product of the root lattice where  $W$  is the Weyl group of the Cartan datum. Let  $\alpha_i, i \in I$  be the positive simple roots, and we define

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad (2.1)$$

$$q_i := q^{\frac{1}{2}(\alpha_i, \alpha_i)} := e^{\pi \mathbf{i} b_i^2}, \quad (2.2)$$

where  $A = (a_{ij})$  is the Cartan matrix. We will let  $\alpha_1$  be the short root in type  $B_n$  and the long root in type  $C_n, F_4$  and  $G_2$ .

We choose

$$\frac{1}{2}(\alpha_i, \alpha_i) = \begin{cases} 1 & i \text{ is long root or in the simply-laced case,} \\ \frac{1}{2} & i \text{ is short root in type } B, C, F, \\ \frac{1}{3} & i \text{ is short root in type } G_2, \end{cases} \quad (2.3)$$

and  $(\alpha_i, \alpha_j) = -1$  when  $i, j$  are adjacent in the Dynkin diagram.

Therefore in the case when  $\mathfrak{g}$  is of type  $B_n, C_n$  and  $F_4$ , if we define  $b_l = b$ , and  $b_s = \frac{b}{\sqrt{2}}$  we have the following normalization:

$$q_i = \begin{cases} e^{\pi i b_l^2} = q & i \text{ is long root,} \\ e^{\pi i b_s^2} = q^{\frac{1}{2}} & i \text{ is short root.} \end{cases} \quad (2.4)$$

In the case when  $\mathfrak{g}$  is of type  $G_2$ , we define  $b_l = b$ , and  $b_s = \frac{b}{\sqrt{3}}$ , and we have the following normalization:

$$q_i = \begin{cases} e^{\pi i b_l^2} = q & i \text{ is long root,} \\ e^{\pi i b_s^2} = q^{\frac{1}{3}} & i \text{ is short root.} \end{cases} \quad (2.5)$$

**Definition 2.2.** Let  $A = (a_{ij})$  denotes the Cartan matrix. Then  $\mathcal{U}_q(\mathfrak{g})$  with  $q = e^{\pi i b_l^2}$  is the algebra generated by  $E_i, F_i$  and  $K_i^{\pm 1}$ ,  $i \in I$  subject to the following relations:

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad (2.6)$$

$$K_i F_j = q_i^{-a_{ij}} F_j K_i, \quad (2.7)$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad (2.8)$$

together with the Serre relations for  $i \neq j$ :

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-k]_{q_i}! [k]_{q_i}!} E_i^k E_j E_i^{1-a_{ij}-k} = 0, \quad (2.9)$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \frac{[1-a_{ij}]_{q_i}!}{[1-a_{ij}-k]_{q_i}! [k]_{q_i}!} F_i^k F_j F_i^{1-a_{ij}-k} = 0, \quad (2.10)$$

where  $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$ .

We choose the Hopf algebra structure of  $\mathcal{U}_q(\mathfrak{g})$  to be given by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i, \quad (2.11)$$

$$\Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes 1, \quad (2.12)$$

$$\Delta(K_i) = K_i \otimes K_i, \quad (2.13)$$

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1, \quad (2.14)$$

$$S(E_i) = -q_i E_i, \quad S(F_i) = -q_i^{-1} F_i, \quad S(K_i) = K_i^{-1}. \quad (2.15)$$

(We will not need the counit and antipode in this paper.)

We define  $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$  to be the real form of  $\mathcal{U}_q(\mathfrak{g})$  induced by the star structure

$$E_i^* = E_i, \quad F_i^* = F_i, \quad K_i^* = K_i. \quad (2.16)$$

Finally, according to the results of [14, 15], we define the modular double  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  to be

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) \quad \mathfrak{g} \text{ is simply-laced}, \quad (2.17)$$

$$\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}}) := \mathcal{U}_q(\mathfrak{g}_{\mathbb{R}}) \otimes \mathcal{U}_{\tilde{q}}({}^L\mathfrak{g}_{\mathbb{R}}) \quad \text{otherwise}, \quad (2.18)$$

where  $\tilde{q} = e^{\pi i b_s^{-2}}$ , and  ${}^L\mathfrak{g}_{\mathbb{R}}$  is the Langlands dual of  $\mathfrak{g}_{\mathbb{R}}$  obtained by interchanging the long roots and short roots of  $\mathfrak{g}_{\mathbb{R}}$ .

## 2.2 Positive representations of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$

In [10, 14, 15], a special class of representations for  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ , called the positive representations, is defined. The generators of the quantum groups are realized by positive essentially self-adjoint operators, and also satisfy the so-called *transcendental relations*, relating the quantum group with its modular double counterpart. More precisely, we have

**Theorem 2.3.** *Let the rescaled generators be*

$$\mathbf{e}_i := 2 \sin(\pi b_i^2) E_i, \quad \mathbf{f}_i := 2 \sin(\pi b_i^2) F_i. \quad (2.19)$$

Note that  $2 \sin(\pi b_i^2) = \left( \frac{i}{q_i - q_i^{-1}} \right)^{-1} > 0$ . Then there exists a family of representations  $\mathcal{P}_{\lambda}$  of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  parametrized by the  $\mathbb{R}_+$ -span of the cone of positive weights  $\lambda \in P_{\mathbb{R}}^+$ , or equivalently by  $\lambda \in \mathbb{R}_+^n$  where  $n = \text{rank}(\mathfrak{g})$ , such that

- The generators  $\mathbf{e}_i, \mathbf{f}_i, K_i$  are represented by positive essentially self-adjoint operators acting on  $L^2(\mathbb{R}^{l(w_0)})$ , where  $l(w_0)$  is the length of the longest element  $w_0 \in W$  of the Weyl group.
- Define the transcendental generators:

$$\tilde{\mathbf{e}}_i := \mathbf{e}_i^{\frac{1}{b_i^2}}, \quad \tilde{\mathbf{f}}_i := \mathbf{f}_i^{\frac{1}{b_i^2}}, \quad \widetilde{K}_i := K_i^{\frac{1}{b_i^2}}. \quad (2.20)$$

Then

- if  $\mathfrak{g}$  is simply-laced, the generators  $\tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i, \widetilde{K}_i$  are obtained by replacing  $b$  with  $b^{-1}$  in the representations of the generators  $\mathbf{e}_i, \mathbf{f}_i, K_i$ .
- If  $\mathfrak{g}$  is of type  $B, C, F, G$ , then the generators  $\widetilde{E}_i, \widetilde{F}_i, \widetilde{K}_i$  with

$$\widetilde{\mathbf{e}}_i := 2 \sin(\pi b_i^{-2}) \widetilde{E}_i, \quad \widetilde{\mathbf{f}}_i := 2 \sin(\pi b_i^{-2}) \widetilde{F}_i \quad (2.21)$$

generate  $\mathcal{U}_{\tilde{q}}({}^L\mathfrak{g}_{\mathbb{R}})$  defined in the previous section.



- The generators  $e_i, f_i, K_i$  and  $\widetilde{e}_i, \widetilde{f}_i, \widetilde{K}_i$  commute weakly up to a sign.

The positive representations are constructed for each reduced expression  $w_0 \in W$  of the longest element of the Weyl group, and representations corresponding to different reduced expressions are unitary equivalent.

**Definition 2.4.** Fix a reduced expression of  $w_0 = s_{i_1} \dots s_{i_N}$ . Let the coordinates of  $L^2(\mathbb{R}^N)$  be denoted by  $\{u_i^k\}$  so that  $i$  is the corresponding root index, and  $k$  denotes the sequence this root is appearing in  $w_0$  from the right. Also denote by  $\{v_j\}_{j=1}^N$  the same set of coordinates counting from the left,  $v(i, k)$  the index such that  $u_i^k = v_{v(i, k)}$ , and  $r(k)$  the root index corresponding to  $v_k$ .

**Example 2.5.** The coordinates of  $L^2(\mathbb{R}^6)$  for  $A_3$  corresponding to  $w_0 = s_3 s_2 s_1 s_3 s_2 s_3$  is given by

$$(u_3^3, u_2^2, u_1^1, u_3^2, u_2^1, u_3^1) = (v_1, v_2, v_3, v_4, v_5, v_6).$$

**Definition 2.6.** We denote by  $p_u = \frac{1}{2\pi i} \frac{\partial}{\partial u}$  and

$$e(u) := e^{\pi b u}, \quad [u] := q^{\frac{1}{2}} e(u) + q^{-\frac{1}{2}} e(-u), \quad (2.22)$$

so that

$$[u]e(-2p) := (q^{\frac{1}{2}} e^{\pi b u} + q^{-\frac{1}{2}} e^{-\pi b u}) e^{-2\pi b p} = e^{\pi b(u-2p)} + e^{\pi b(-u-2p)} \quad (2.23)$$

is positive whenever  $[p, u] = \frac{1}{2\pi i}$ .

(Note that we changed the notation slightly for  $e(u)$  in this paper from previous references [14]-[16] for later convenience.)

**Definition 2.7.** By abuse of notation, we denote by

$$[u_s + u_l]e(-2p_s - 2p_l) := e^{\pi b_s(-u_s - 2p_s) + \pi b_l(-u_l - 2p_l)} + e^{\pi b_s(u_s - 2p_s) + \pi b_l(u_l - 2p_l)}, \quad (2.24)$$

where  $u_s$  (resp.  $u_l$ ) is a linear combination of the variables corresponding to short roots (resp. long roots). The parameters  $\lambda_i$  are also considered in both cases. Similarly  $p_s$  (resp.  $p_l$ ) are linear combinations of the  $p$  shifting of the short roots (resp. long roots) variables. This applies to all simple  $\mathfrak{g}$ , with the convention given in Definition 2.1.

We will occasionally write in the form  $[X]e(-2p_Y)$ , where by abuse of notation, if  $Y = \sum a_i u_i$ ,  $a_i \in \mathbb{Z}$ , then  $p_Y = \sum a_i p_{u_i}$ .

**Theorem 2.8.** [14, 15] For a fixed reduced expression of  $w_0$ , the positive representation  $\mathcal{P}_\lambda$  is given by

$$f_i = \sum_{k=1}^n \left[ - \sum_{j=1}^{v(i, k)-1} a_{i, r(j)} v_j - u_i^k - 2\lambda_i \right] e(2p_i^k), \quad (2.25)$$

$$K_i = e \left( - \sum_{k=1}^{l(w_0)} a_{i, r(k)} v_k + 2\lambda_i \right). \quad (2.26)$$

By taking  $w_0 = w's_i$  so that the simple reflection for root  $i$  appears on the right, the action of  $e_i$  is given by

$$e_i = [u_i^1]e(-2p_i^1). \quad (2.27)$$

In this paper, it is instructive to recall the explicit expression in the case of rank 1 and 2. For details of the construction and the other cases please refer to [14, 15].

**Proposition 2.9.** [1, 26] *The positive representation  $P_\lambda$  of  $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(2, \mathbb{R}))$  acting on  $L^2(\mathbb{R})$  is given by*

$$\begin{aligned} e &= [u - \lambda]e(-2p) = e^{\pi b(-u + \lambda - 2p)} + e^{\pi b(u - \lambda - 2p)}, \\ f &= [-u - \lambda]e(2p) = e^{\pi b(u + \lambda + 2p)} + e^{\pi b(-u - \lambda + 2p)}, \\ K &= e(-2u) = e^{-2\pi bu}. \end{aligned}$$

(Note that it is unitary equivalent to the canonical form (2.25)-(2.27) by  $u \mapsto u + \lambda$ .)

**Proposition 2.10.** [14] *The positive representation  $\mathcal{P}_\lambda$  of  $\mathcal{U}_{q\bar{q}}(\mathfrak{sl}(3, \mathbb{R}))$  with parameters  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ , corresponding to the reduced expression  $w_0 = s_2 s_1 s_2$ , acting on  $f(u, v, w) \in L^2(\mathbb{R}^3)$ , is given by*

$$\begin{aligned} e_1 &= [v - w]e(-2p_v) + [u]e(-2p_v + 2p_w - 2p_u), \\ e_2 &= [w]e(-2p_w), \\ f_1 &= [-v + u - 2\lambda_1]e(2p_v), \\ f_2 &= [-2u + v - w - 2\lambda_2]e(2p_w) + [-u - 2\lambda_2]e(2p_u), \\ K_1 &= e(u - 2v + w - 2\lambda_1), \\ K_2 &= e(-2u + v - 2w - 2\lambda_2). \end{aligned}$$

**Proposition 2.11.** [15] *The positive representation  $\mathcal{P}_\lambda$  of  $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$  with parameters  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ , where  $\mathfrak{g}_{\mathbb{R}}$  is of type  $B_2$ , corresponding to the reduced expression  $w_0 = s_1 s_2 s_1 s_2$ , acting on  $f(t, u, v, w) \in L^2(\mathbb{R}^4)$ , is given by*

$$\begin{aligned} e_1 &= [t]e(-2p_t - 2p_u + 2p_w) + [u - v]e(-2p_u - 2p_v + 2p_w) + [v - w]e(-2p_v), \\ e_2 &= [w]e(-2p_w), \\ f_1 &= [-t - 2\lambda_1]e(2p_t) + [-2t + u - v - 2\lambda_1]e(2p_v), \\ f_2 &= [2t - u - 2\lambda_2]e(2p_u) + [2t - 2u + 2v - w - 2\lambda_2]e(2p_w), \\ K_1 &= e(-2t + u - 2v + w - 2\lambda_1), \\ K_2 &= e(2t - 2u + 2v - 2w - 2\lambda_2). \end{aligned}$$

In this case (cf. Definition 2.7),  $u_s$  is linear combinations of  $\{t, v\}$ , while  $u_l$  is linear combinations of  $\{u, w\}$ . Similarly for  $p_s$  and  $p_l$ .

We will omit the case of type  $G_2$  for simplicity.

### 2.3 Quantum dilogarithm $G_b(x)$ and $g_b(x)$

First introduced by Faddeev and Kashaev [4, 6], the quantum dilogarithm  $G_b(x)$  and its variants  $g_b(x)$  play a crucial role in the study of positive representations of split real quantum groups, and also appear in many other areas of mathematics and physics, most notably cluster algebras, quantum Teichmüller theory and Liouville CFT. In this subsection, let us recall the definition and some properties of the quantum dilogarithm functions [1, 13, 26] that is needed in the calculations of this paper.

**Definition 2.12.** *The quantum dilogarithm function  $G_b(x)$  is defined on  $0 \leq \operatorname{Re}(z) \leq Q = b + b^{-1}$  by*

$$G_b(x) = \overline{\zeta}_b \exp \left( - \int_{\Omega} \frac{e^{\pi t z}}{(e^{\pi b t} - 1)(e^{\pi b^{-1} t} - 1)} \frac{dt}{t} \right), \quad (2.28)$$

where

$$\zeta_b = e^{\frac{\pi i}{2} \left( \frac{b^2 + b^{-2}}{6} + \frac{1}{2} \right)}, \quad (2.29)$$

and the contour goes along  $\mathbb{R}$  with a small semicircle going above the pole at  $t = 0$ . This can be extended meromorphically to the whole complex plane with poles at  $x = -nb - mb^{-1}$  and zeros at  $x = Q + nb + mb^{-1}$ , for  $n, m \in \mathbb{Z}_{\geq 0}$ .

We also define the function  $S_b(x)$  as follows.

**Definition 2.13.** *The function  $S_b(x)$  is defined by*

$$S_b(x) := e^{\frac{\pi i}{2} x(Q-s)} G_b(x). \quad (2.30)$$

The quantum dilogarithm satisfies the following properties:

**Proposition 2.14.** *Self-duality:*

$$S_b(x) = S_{b^{-1}}(x), \quad G_b(x) = G_{b^{-1}}(x); \quad (2.31)$$

*Functional equations:*

$$S_b(x + b^{\pm 1}) = i(e^{-\pi i b^{\pm 1} x} - e^{\pi i b^{\pm 1} x}) S_b(x), \quad G_b(x + b^{\pm 1}) = (1 - e^{2\pi i b^{\pm 1} x}) G_b(x); \quad (2.32)$$

*Reflection property:*

$$S_b(x) S_b(Q - x) = 1, \quad G_b(x) G_b(Q - x) = e^{\pi i x(Q-Q)}; \quad (2.33)$$

*Complex conjugation:*

$$\overline{S_b(x)} = \frac{1}{S_b(Q - \bar{x})}, \quad \overline{G_b(x)} = \frac{1}{G_b(Q - \bar{x})}, \quad (2.34)$$

in particular

$$\left| S_b\left(\frac{Q}{2} + ix\right) \right| = \left| G_b\left(\frac{Q}{2} + ix\right) \right| = 1 \text{ for } x \in \mathbb{R}; \quad (2.35)$$

Asymptotic properties:

$$G_b(x) \sim \begin{cases} \bar{\zeta}_b & \text{Im}(x) \rightarrow +\infty \\ \zeta_b e^{\pi i x(x-Q)} & \text{Im}(x) \rightarrow -\infty \end{cases}. \quad (2.36)$$

**Lemma 2.15** (*q*-binomial theorem). *For positive self-adjoint variables  $U, V$  with  $UV = q^2 VU$ , we have:*

$$(U + V)^{ib^{-1}t} = \int_C \left( \begin{matrix} it \\ i\tau \end{matrix} \right)_b U^{ib^{-1}(t-\tau)} V^{ib^{-1}\tau} d\tau, \quad (2.37)$$

where the *q*-beta function (or *q*-binomial coefficient) is given by

$$\left( \begin{matrix} t \\ \tau \end{matrix} \right)_b = \frac{G_b(-\tau)G_b(\tau-t)}{G_b(-t)}, \quad (2.38)$$

and  $C$  is the contour along  $\mathbb{R}$  that goes above the pole at  $\tau = 0$  and below the pole at  $\tau = t$ .

**Lemma 2.16** (tau-beta theorem). *We have*

$$\int_C e^{-2\pi\tau\beta} \frac{G_b(\alpha + i\tau)}{G_b(Q + i\tau)} d\tau = \frac{G_b(\alpha)G_b(\beta)}{G_b(\alpha + \beta)}, \quad (2.39)$$

where the contour  $C$  goes along  $\mathbb{R}$  and goes above the poles of  $G_b(Q + i\tau)$  and below those of  $G_b(\alpha + i\tau)$ . By the asymptotic properties of  $G_b$ , the integral converges for  $\text{Re}(\beta) > 0, \text{Re}(\alpha + \beta) < Q$ .

We will also need another important variant of the quantum dilogarithm.

**Definition 2.17.** *The function  $g_b(x)$  is defined by*

$$g_b(x) = \frac{\bar{\zeta}_b}{G_b\left(\frac{Q}{2} + \frac{\log x}{2\pi ib}\right)}, \quad (2.40)$$

where  $\log$  takes the principal branch of  $x$ .

**Lemma 2.18.** *[1, (3.31), (3.32)] We have the following Fourier transformation formula:*

$$\int_{\mathbb{R}+i0} \frac{e^{-\pi it^2}}{G_b(Q + it)} X^{ib^{-1}t} dt = g_b(X), \quad (2.41)$$

$$\int_{\mathbb{R}+i0} \frac{e^{-\pi Qt}}{G_b(Q + it)} X^{ib^{-1}t} dt = g_b^*(X), \quad (2.42)$$

where  $X$  is a positive operator and the contour goes above the pole at  $t = 0$ .

We will need the following properties of  $g_b(x)$ .

**Lemma 2.19.** *By (2.35),  $|g_b(x)| = 1$  when  $x \in \mathbb{R}_+$ , hence  $g_b(X)$  is a unitary operator for any positive operator  $X$ . Furthermore, by (2.31) and Lemma 2.18, we have the self-duality of  $g_b(x)$  given by*

$$g_b(X) = g_{b^{-1}}(X^{\frac{1}{b^2}}). \quad (2.43)$$

**Lemma 2.20.** *If  $UV = q^2VU$  where  $U, V$  are positive self adjoint operators, then*

$$g_b(U)g_b(V) = g_b(U + V), \quad (2.44)$$

$$g_b(U)^*Vg_b(U) = q^{-1}UV + V, \quad (2.45)$$

$$g_b(V)Ug_b(V)^* = U + q^{-1}UV. \quad (2.46)$$

Note that (2.44) and (2.45) together imply the pentagon relation

$$g_b(V)g_b(U) = g_b(U)g_b(q^{-1}UV)g_b(V). \quad (2.47)$$

If  $UV = q^4VU$ , then we apply the Lemma twice and obtain

$$g_b(U)^*Vg_b(U) = V + [2]_q q^2VU + q^4VU^2, \quad (2.48)$$

$$g_b(V)Ug_b(V)^* = U + [2]_q q^{-2}UV + q^{-4}UV^2. \quad (2.49)$$

where  $[2]_q = q + q^{-1}$ .

As a consequence of the above Lemma, we also have the following

**Lemma 2.21.** *[15, 33] If  $UV = q^2VU$  where  $U, V$  are positive essentially self-adjoint operators, then  $U + V$  is positive essentially self-adjoint, and*

$$(U + V)^{\frac{1}{b^2}} = U^{\frac{1}{b^2}} + V^{\frac{1}{b^2}}. \quad (2.50)$$

## 2.4 Lusztig's isomorphism

We recall several definitions and results concerning the Lusztig's isomorphism in the positive representation setting that is needed in this paper. Refer to [16] for more details.

Denote by

$$[u, v]_q := quv - q^{-1}vu. \quad (2.51)$$

Fix a positive representation  $\mathcal{P}_\lambda$ .

**Proposition 2.22.** *In the simply-laced case, the operators*

$$e_{ij} := \frac{[e_j, e_i]_{q^{\frac{1}{2}}}}{q - q^{-1}} = \frac{q^{\frac{1}{2}}e_j e_i - q^{-\frac{1}{2}}e_i e_j}{q - q^{-1}} \quad (2.52)$$

*are positive essentially self-adjoint.*

**Proposition 2.23.** *In the non-simply-laced case, the operators*

$$e_{ij} = (-1)^{a_{ij}} \left[ \left[ e_i, \dots [e_i, e_j]_{q_i^{\frac{a_{ij}}{2}}} \right]_{q_i^{\frac{a_{ij}+2}{2}}} \dots \right]_{q_i^{\frac{-a_{ij}-2}{2}}} \prod_{k=1}^{-a_{ij}} (q_i^k - q_i^{-k})^{-1} \quad (2.53)$$

*are positive essentially self-adjoint.*

**Theorem 2.24.** *For any root  $i \in I$ , there exists a unitary operator  $T_i$  such that*

$$T_i(e_i) = q_i f_i K_i^{-1} = q_i^{-1} K_i^{-1} f_i, \quad (2.54)$$

$$T_i(f_i) = q_i^{-1} K_i e_i = q_i e_i K_i, \quad (2.55)$$

$$T_i(e_j) = e_{ij}, \text{ for } i, j \text{ adjacent}, \quad (2.56)$$

$$T_i(f_j) = f_{ij}, \text{ for } i, j \text{ adjacent}, \quad (2.57)$$

$$T_i(e_k) = e_k, \text{ for } a_{ik} = 0, \quad (2.58)$$

$$T_i(f_k) = f_k, \text{ for } a_{ik} = 0, \quad (2.59)$$

$$T_i(K_j) = K_j K_i^{-a_{ij}}. \quad (2.60)$$

**Proposition 2.25.** *[22, 23] The operators  $T_i$  satisfy the Coxeter relations:*

$$\underbrace{T_i T_j T_i \dots}_{-a'_{ij}+2} = \underbrace{T_j T_i T_j \dots}_{-a'_{ji}+2}, \quad (2.61)$$

where  $-a'_{ij} = \max\{-a_{ij}, -a_{ji}\}$ . Furthermore, for  $\alpha_i, \alpha_j$  simple roots, and an element  $w = s_{i_1} \dots s_{i_k} \in W$  such that  $w(\alpha_i) = \alpha_j$ , we have for  $X = e, f, K$ :

$$T_{i_1} \dots T_{i_k}(X_i) = X_j. \quad (2.62)$$

Without loss of generality, in this paper we will use the inverse version of the following definition.

**Definition 2.26.** *Let  $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$  be the longest element of the Weyl group. We define*

$$e_{\alpha_k} := T_{i_1}^{-1} T_{i_2}^{-1} \dots T_{i_{k-1}}^{-1} e_{i_k}. \quad (2.63)$$

*These are all positive essentially self-adjoint.*

Finally we record some commutation relations among the non-simple generators.

**Proposition 2.27.** *For the simply-laced case, if  $a_{ij} = -1$ , we have*

$$e_i e_{ij} = q^{-1} e_{ij} e_i, \quad (2.64)$$

$$e_j e_{ij} = q e_{ij} e_j, \quad (2.65)$$

$$e_i e_j^n = q^n e_j^n e_i + q^{\frac{1}{2}} (q^{-n} - q^n) e_j^{n-1} e_{ij}. \quad (2.66)$$

**Proposition 2.28.** *For doubly-laced case, let  $a_{ij} = -2$ . Denote  $e_{ij} := T_i e_j$  and  $e_X := T_i T_j e_i$ . We have*

$$e_i^n e_j = q^{-n} e_j e_i^n + q^{\frac{1}{2}}(q^n - q^{-n}) e_i^{n-1} e_X, \quad (2.67)$$

$$e_i^n e_X = e_X e_i^n + q^{\frac{1}{2}}(q^{-n} - 1) e_{ij} e_i^{n-1}, \quad (2.68)$$

$$e_i e_{ij} = q^{-1} e_{ij} e_i, \quad (2.69)$$

$$e_{ij}^n e_j = q^n e_j e_{ij} + (q^{-2n} - 1) e_X^2, \quad (2.70)$$

$$e_{ij} e_X = q^{-1} e_X e_{ij}, \quad (2.71)$$

$$e_X e_j = q^{-1} e_j e_X. \quad (2.72)$$

Note that by means of [16, Proposition 6.8], we can take  $n$  to be a complex power.

## 2.5 GNS representation, multiplicative unitary and multiplier Hopf algebra

Finally, we recall the definition of Gelfand-Naimark-Segal (GNS) representation of a  $C^*$ -algebra, and its corresponding multiplicative unitary that is needed in this paper. These are fundamental to the theory of locally compact quantum groups in the setting of  $C^*$ -algebras and von Neumann algebras, and to the generalization of Pontryagin duality. In particular, a multiplicative unitary encodes all the structure maps of a quantum group and its dual. See [13], [21] or [31] for more discussions.

**Definition 2.29.** *A Gelfand-Naimark-Segal (GNS) representation of a  $C^*$ -algebra  $\mathcal{A}$  with a weight  $\phi$  is a triple*

$$(\mathcal{H}, \pi, \Lambda),$$

where  $\mathcal{H}$  is a Hilbert space,  $\Lambda : \mathcal{A} \rightarrow \mathcal{H}$  is a linear map, and  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  is a representation of  $\mathcal{A}$  on  $\mathcal{H}$  such that  $\Lambda(\mathcal{N})$  is dense in  $\mathcal{H}$ , and

$$\pi(a)\Lambda(b) = \Lambda(ab) \quad \forall a \in \mathcal{A}, b \in \mathcal{N}, \quad (2.73)$$

$$\langle \Lambda(a), \Lambda(b) \rangle = \phi(b^*a) \quad \forall a, b \in \mathcal{N}, \quad (2.74)$$

where  $\mathcal{N} = \{a \in \mathcal{A} : \phi(a^*a) < \infty\}$ .

**Definition 2.30.** *Let  $\mathcal{H}$  be a Hilbert space. A unitary operator  $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$  is called a multiplicative unitary if it satisfies the pentagon equation*

$$W_{23}W_{12} = W_{12}W_{13}W_{23}, \quad (2.75)$$

where the standard leg notation is used.

Given a GNS representation  $(\mathcal{H}, \pi, \Lambda)$  of a locally compact quantum group  $\mathcal{A}$ , we can define a unitary operator

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)). \quad (2.76)$$

It is known that  $W$  is a multiplicative unitary [21, Thm 3.16, Thm 3.18], and the coproduct on  $\mathcal{A}$  defining it can be recovered from  $W$ :

**Proposition 2.31.** *Let  $x \in \mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$  as operator. Then*

$$W^*(1 \otimes x)W = \Delta(x) \quad (2.77)$$

*as operators on  $\mathcal{H} \otimes \mathcal{H}$ .*

*Proof.* [13] For  $x, f, g \in \mathcal{A}$ , we have  $x \cdot \Lambda(f) = \Lambda(xf)$ , hence

$$\begin{aligned} W^*((1 \otimes x) \cdot (\Lambda(f) \otimes \Lambda(g))) &= W^*(\Lambda(f) \otimes \Lambda(xg)) \\ &= (\Lambda \otimes \Lambda)(\Delta(xg)(f \otimes 1)) \\ &= (\Lambda \otimes \Lambda)(\Delta(x)\Delta(g)(f \otimes 1)) \\ &= \Delta(x) \cdot (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1)) \\ &= \Delta(x)W^*(f \otimes g). \end{aligned}$$

□

In particular, Proposition 2.31 provides a unitary equivalence

$$W : \mathcal{H} \otimes \mathcal{H} \simeq M \otimes \mathcal{H} \quad (2.78)$$

of the tensor product of the GNS representations, where  $M$  is the multiplicity module equipped with the trivial action. This is the key step to proving the tensor product decomposition of the positive representations of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ .

Finally, we recall the definition of a multiplier Hopf algebra.

**Definition 2.32.** *Let  $\mathcal{B}(\mathcal{H})$  be the algebra of bounded linear operators on a Hilbert space  $\mathcal{H}$ . Then the multiplier algebra  $M(\mathcal{A})$  of a  $C^*$ -algebra  $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$  is the  $C^*$ -algebra of operators*

$$M(\mathcal{A}) = \{b \in \mathcal{B}(\mathcal{H}) : b\mathcal{A} \subset \mathcal{A}, \mathcal{A}b \subset \mathcal{A}\}. \quad (2.79)$$

*In particular,  $\mathcal{A}$  is an ideal of  $M(\mathcal{A})$ .*

**Definition 2.33.** *A multiplier Hopf  $*$ -algebra is a  $C^*$ -algebra  $\mathcal{A}$  together with the antipode  $S$ , the counit  $\epsilon$ , and the coproduct map*

$$\Delta : \mathcal{A} \longrightarrow M(\mathcal{A} \otimes \mathcal{A}), \quad (2.80)$$

*all of which can be extended to a map from  $M(\mathcal{A})$ , such that the usual properties of a Hopf algebra holds on the level of  $M(\mathcal{A})$ .*

### 3 Positive representations restricted to Borel subalgebra

In the case of  $\mathfrak{g} = \mathfrak{sl}_2$ , the Borel subalgebra  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  is generated by the operator  $K$  and  $\mathbf{e}$  satisfying the quantum plane relation

$$K\mathbf{e} = q^2\mathbf{e}K, \quad (3.1)$$



in the sense of integrable representations [29], i.e.

$$K^{ib^{-1}s} \mathbf{e}^{ib^{-1}t} = e^{-2\pi i s t} \mathbf{e}^{ib^{-1}t} K^{ib^{-1}s}, \quad \forall s, t \in \mathbb{R} \quad (3.2)$$

as unitary operators. Thus the positive representation  $\mathcal{P}_\lambda \simeq L^2(\mathbb{R})$  restricted to the Borel part is nothing but an irreducible representation of the quantum plane, such that the operators are represented by positive self-adjoint operators. By Stone-von Neumann's Theorem, we know that every such representation is unitary equivalent to the canonical representation  $\mathcal{H} \simeq L^2(\mathbb{R})$  given by

$$K = e^{-2\pi b x}, \quad \mathbf{e} = e^{-2\pi b p}. \quad (3.3)$$

In particular, the representation does not depend on any parameters. In fact, from Proposition 2.9, it is easily seen that  $\Phi : \mathcal{P}_\lambda \longrightarrow \mathcal{H}$  given by

$$\Phi = e^{-\frac{\pi i(x-\lambda)^2}{2}} g_b(e^{-2\pi b(x-\lambda)}) \quad (3.4)$$

is the required unitary equivalence.

In this section, we prove that the positive representation  $\mathcal{P}_\lambda$  of any  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to the Borel subalgebra in fact also does not depend on the parameter  $\lambda$ .

**Theorem 3.1.** *We have*

$$\mathcal{P}_\lambda \simeq \mathcal{P}_{\lambda'} \quad (3.5)$$

as representations of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  for any  $\lambda, \lambda' \in \mathbb{R}^{\text{rank}(\mathfrak{g})}$

*Proof.* From [14, Section 11] for the simply-laced case, or [15, Section 9] for non-simply-laced case, we know that

$$\mathcal{P}_\lambda \simeq \mathcal{P}_{w(\lambda)},$$

where  $w \in W$  is any Weyl group element acting on  $\lambda$ , namely for simple reflections,

$$s_i(\lambda_j) := \lambda_j - a_{ij}\lambda_i,$$

where  $a_{ij}$  is the Cartan matrix. In particular, this means that for each root  $i \in I$ , there exists a unitary transformation  $B_i$  such that it fixes the action of  $\mathbf{e}_i$ , but modify the action of  $K_j$  by

$$K_j \mapsto K_j e^{a_{ij}\pi b \lambda_i}.$$

However, when we restrict to the Borel subalgebra, we are released from the restriction of the action of the lower Borel generated by  $\mathbf{f}_i$ , in which the proof above allows us to define the unitary transformation  $B_i$  such that it fixes the action of  $\mathbf{e}_i$  but modify the action of  $K_i$  by an arbitrary constant  $c_i$ :

$$K_j \mapsto K_j e^{a_{ij}\pi b c_i}.$$

Therefore, it suffices to solve the equation  $\sum a_{ij}c_i = \lambda_i$  in order for all  $\lambda_i$  in the representation to cancel. This clearly can be done because the Cartan matrix is invertible. In particular, all representations  $\mathcal{P}_\lambda$  are unitary equivalent to  $\mathcal{P}_{\vec{0}}$ .  $\square$

Let us denote by  $\mathcal{P} := \mathcal{P}_{\vec{\partial}}$ . We make the following conjecture:

**Conjecture 3.2.** *Any irreducible representations of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  satisfying the conditions of a positive representation is unitary equivalent to  $\mathcal{P}$ . This generalizes the Stone von-Neumann Theorem to the case of Borel subalgebra of a split real quantum group. More specifically, using the notion of “integrable representations” of quantum plane [29, 13], we require as unitary operators,*

$$K_i^{\mathfrak{b}^{-1}s} e_j^{\mathfrak{b}^{-1}t} = e^{-\pi a_{ij}st} e_j^{\mathfrak{b}^{-1}t} K_i^{\mathfrak{b}^{-1}s}$$

and in the simply-laced case,

$$e_i^{\mathfrak{b}^{-1}s} e_{ij}^{\mathfrak{b}^{-1}t} = e^{\pi st} e_{ij}^{\mathfrak{b}^{-1}t} e_i^{\mathfrak{b}^{-1}s},$$

$$e_j^{\mathfrak{b}^{-1}s} e_{ij}^{\mathfrak{b}^{-1}t} = e^{-\pi st} e_{ij}^{\mathfrak{b}^{-1}t} e_j^{\mathfrak{b}^{-1}s},$$

where the latter two equations correspond to the Serre’s relation. We have similar equations for the non-simply-laced case (cf. Proposition 2.28).

**Remark 3.3.** *In fact, it is known that, at least for type  $A_n$ , the representation with all  $[X]e(-2p_Y)$  in the expression of  $\mathcal{P}_{\lambda}$  replaced by  $e(X - 2p_Y)$  is unitary equivalent to  $\mathcal{P}_{\lambda}$ . The general algorithm is not yet known, but it hints at the conjecture. We will use this result in the explicit computation of type  $A_2$  and  $A_3$  in Section 6 of this paper.*

## 4 Positive representations from GNS representation of $\mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$

In this section, we recall the definition of  $\mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$  and construct the GNS-representation by left multiplication.

### 4.1 Definition of $\mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$

In [16], we used the language of multiplier Hopf algebra, and defined a  $C^*$ -algebraic version of the Borel subalgebra using a continuous basis. Explicitly, consider the action of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  on  $\mathcal{P}_{\lambda} \simeq \mathcal{P} \simeq L^2(\mathbb{R}^N)$ . where  $N = l(w_0)$ ,  $n = \text{rank}(\mathfrak{g})$ .

**Definition 4.1.** *We define the  $C^*$ -algebraic version of the Borel subalgebra*

$$Ub := \mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$$

as the operator norm closure of the linear span of all bounded operators on  $L^2(\mathbb{R}^N)$  of the form

$$\vec{F} := \left( \prod_{k=1}^N \int_C \frac{F_k(t_k)}{G_{b_{i_k}}(Q_{i_k} + \dot{t}_k)} e_{\alpha_k}^{\mathfrak{b}^{-1}t_k} dt_k \right) \left( \prod_{i=1}^n \int_{\mathbb{R}} \widehat{F}_i(s_i) K_i^{\mathfrak{b}^{-1}s_i} ds_i \right), \quad (4.1)$$

where  $e_{\alpha_k}$  is given by (2.63) with the order of multiplication as  $e_{\alpha_1} \dots e_{\alpha_N}$ ,  $F_k(t_k)$  and  $\widehat{F}_i(s_i)$  are both entire analytic functions that have rapid decay along the real direction (i.e. for fixed  $y_0$ ,  $F_k(x + iy_0)$  decays faster than any exponential function in  $x$ ). Finally the contour  $C$  is along the real axis which goes above the pole of  $G_b$  at  $t_k = 0$ .

It was shown in [16] that  $\mathbf{Ub}$  is a multiplier Hopf alger (cf. Definition 2.33).

**Remark 4.2.** We modified the definition slightly from [16]. We have used the opposite multiplication order to be consistent with the definition of  $e_{\alpha_k}$  using  $T_{i_k}^{-1}$  (cf. (2.63)) instead of  $T_{i_k}$ . The two definitions are identical. Moreover, to simplify calculations, we have used the functions  $\widehat{F}_i(s_i)$  parametrized by the powers of  $K_i$  instead of a general compact function  $F_0(\mathbf{H})$  considered in [16], where  $K_i = q_i^{H_i}$ . These are simply related by Fourier transform  $\mathcal{F}$ :

$$\int (\mathcal{F}\widehat{F})(s) K^{ib^{-1}s} ds = \iint e^{-2\pi i s t} \widehat{F}(t) K^{ib^{-1}s} dt ds = \widehat{F}\left(\frac{1}{2}ibH\right).$$

## 4.2 GNS-representation

It turns out the definition above does not necessarily define a positive action. In order to obtain an action of  $\mathcal{U}_{q\bar{q}}(\beta_{\mathbb{R}})$  that is positive, we modify the expression of  $\vec{F} \in \mathcal{A}$  as follows.

**Definition 4.3.** We define the GNS representation  $(\mathcal{H}, \pi, \Lambda)$  of  $\mathcal{A} := \mathcal{U}_{q\bar{q}}^{C*}(\mathfrak{b}_{\mathbb{R}})$  on  $\mathcal{H} \simeq L^2(\mathbb{R}^{N+n})$  as follows. We let

$$\vec{F} := \prod_{k=1}^N \left( \int_C \frac{F_k(t_k)}{S_{b_{i_k}}(Q_{i_k} + it_k)} e^{ib_{i_k}^{-1}t_k} dt_k K_{\alpha_k}^{\frac{Q}{2b}} \right) \left( \prod_{i=1}^n \int_{\mathbb{R}} \widehat{F}_i(s_i) K_i^{ib^{-1}s_i} e^{\pi Q s_i} ds_i \right), \quad (4.2)$$

and define the representation

$$\begin{aligned} \Lambda : \mathcal{A} &\longrightarrow \mathcal{H} \\ \vec{F} &\mapsto \prod_{k=1}^N F_k(t_k) \prod_{i=1}^n \widehat{F}_i(s_i) \end{aligned} \quad (4.3)$$

and extend by linearity. Finally, the  $L^2$ -norm of  $\mathcal{H}$  (cf. [13, Theorem 4.9]) will be defined for each variable as

$$\|F(\dots, t_k, \dots)\|^2 := \int_{\mathbb{R}^N} \left| F\left(\dots, t_k + \frac{iQ_{i_k}}{2}, \dots\right) \right|^2 dt_1 \dots dt_N, \quad (4.4)$$

$$\|\widehat{F}(\dots, s_i, \dots)\|^2 := \int_{\mathbb{R}^n} |\widehat{F}(\dots, s_i, \dots)|^2 ds_1 \dots ds_n. \quad (4.5)$$

Note that we have used  $S_b(x)$  instead in the denominator. Also we slipped  $K_{\alpha_k}$  in between the products of  $\mathbf{e}_{\alpha_k}$ , this is well defined since  $K_{\frac{Q}{2b}}$  is in the multiplier  $M(\mathcal{A})$  of  $\mathcal{A}$ . Finally note that the last definition of the  $L^2$ -norm for  $F_k(t_k)$  can be thought of as shifting the arguments by  $\frac{iQ}{2}$  and use the usual  $L^2$ -norm. More precisely,

$$\begin{aligned} \mathcal{H} &\longrightarrow L^2(\mathbb{R}^{N+n}, dt_1 \dots dt_N ds_1 \dots ds_n) \\ F(\dots, t_k, \dots) &\mapsto F'(\dots, t_k, \dots) := F\left(\dots, t_k + \frac{iQ_{i_k}}{2}, \dots\right) \end{aligned} \quad (4.6)$$

is an isomorphism of Hilbert space. Therefore, to obtain a representation on the usual  $L^2$ -norm, we work on  $F$ , and at the end shift all the variables  $t_k$  by this map.

**Proposition 4.4.** *The GNS representation is unitary equivalent to a representation that is independent on the  $s_i$  variables.*

*Proof.* We note that the action of multiplication by  $\mathbf{e}_i$  from the left will not depend on  $s_i$ . Moreover, multiplication of  $K_i$  from the left, after commuting with all the  $\mathbf{e}_{\alpha_k}$ , give us an extra action factor  $e^{-2\pi b p_{s_i}}$ , which is independent from the  $t_i$  variables. Therefore these  $p_{s_i}$  can be treated as constants  $\lambda_i$ . Once we have shown that the  $t_k$  part is unitary equivalent to the positive representation  $\mathcal{P}$  in Section 5, then by Proposition 3.1 there exists a unitary transformation that can remove the dependence on  $s_i$  completely.  $\square$

In particular the GNS representation is unitary equivalent to

$$\mathcal{H} = L^2(\mathbb{R}^{N+n}) \simeq \mathcal{P} \otimes M_K \quad (4.7)$$

for a trivial multiplicity module  $M_K \simeq L^2(\mathbb{R}^n)$ . Therefore without loss of generality, we will ignore the  $K_i^{ib^{-1}s_i}$  from now on, and the GNS representation calculated below will assume to act trivially on the  $s_i$  variables. Note that the  $\mathbf{e}_{\alpha_k}$  part already gives the correct functional dimension.

### 4.3 Rank 2 case

We illustrate the representation in the rank 2 case, which is essential to generalize to the higher rank. Let us first consider the simply-laced case, and fix the longest element to be  $w_0 = s_2 s_1 s_2$ . Also using Definition 2.4 and Definition 4.1, we name the variables

$$(u, v, w) := (v_1, v_2, v_3) = (t_3, t_2, t_1) = (u_2^2, u_1^1, u_2^1).$$

Then a typical element of  $\vec{F} \in \mathcal{A}$  (ignoring the ending  $K_i$  terms) is given by

$$\begin{aligned}\vec{F} &= \int_C F(u, v, w) \frac{\mathbf{e}_2^{ib^{-1}w} K_2^{\frac{Q}{2b}}}{S_b(Q + \mathbf{i}w)} \frac{\mathbf{e}_{12}^{ib^{-1}v} (K_1 K_2)^{\frac{Q}{2b}}}{S_b(Q + \mathbf{i}v)} \frac{\mathbf{e}_1^{ib^{-1}u} K_1^{\frac{Q}{2b}}}{S_b(Q + \mathbf{i}u)} dudvdw \\ &= \int_C F(u, v, w) e^{-\frac{\pi Q v}{2}} \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w)} \frac{\mathbf{e}_{12}^{ib^{-1}v}}{S_b(Q + \mathbf{i}v)} \frac{\mathbf{e}_1^{ib^{-1}u}}{S_b(Q + \mathbf{i}u)} dudvdw\end{aligned}\quad (4.8)$$

where in the last line we moved the  $K_i$  to the right and absorb into the (ignored)  $K$  terms. Note that we have reversed the order of the variables since we used the opposite multiplication order (reading from left to right).

Let us illustrate how to find the actions of the generators  $\mathbf{e}_i$  in general. The action of  $\mathbf{e}_2$  is easy. It only shifts the variables  $w$ . We have (ignoring other variables)

$$\begin{aligned}\mathbf{e}_2 \cdot \vec{F} &= \int F(u, v, w) \dots \frac{\mathbf{e}_2^{ib^{-1}w+1}}{S_b(Q + \mathbf{i}w)} \dots dw \\ &= \int F(u, v, w + \mathbf{i}b) \dots \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w - b)} \dots dw \\ &= \int i(e^{-\pi \mathbf{i}b(Q + \mathbf{i}w - b)} - e^{\pi \mathbf{i}b(Q + \mathbf{i}w - b)}) F(u, v, w + \mathbf{i}b) \dots \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w)} \dots dw \\ &= -i \int (e^{\pi b w} - e^{-\pi b w}) F(u, v, w + \mathbf{i}b) \dots \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w)} \dots dw,\end{aligned}$$

where we have used property (2.32) of  $S_b$ . Hence the action of  $\mathbf{e}_2$  on  $\mathcal{H}$  (i.e. on the coefficient function) is given by

$$\mathbf{e}_2 = -\mathbf{i}(e^{\pi b w} - e^{-\pi b w})e^{-2\pi b p_w}. \quad (4.9)$$

Finally, the shift  $w \mapsto \frac{\mathbf{i}Q}{2}$  gives us the positive expression on  $L^2(\mathbb{R}^3)$ :

$$\begin{aligned}\mathbf{e}_2 &= -\mathbf{i}(e^{\pi b(w + \frac{\mathbf{i}Q}{2})} - e^{-\pi b(w + \frac{\mathbf{i}Q}{2})})e^{-2\pi b p_w} \\ &= -\mathbf{i}(iq^{\frac{1}{2}}e^{\pi b w} + iq^{-\frac{1}{2}}e^{-\pi b w})e^{-2\pi b p_w} \\ &= e(w - 2p_w) + e(-w - 2p_w).\end{aligned}\quad (4.10)$$

For the action of  $\mathbf{e}_1$ , we used the commutation relations from Proposition 2.27 and proceed as above.

$$\begin{aligned}\mathbf{e}_1 \mathbf{e}_2^{ib^{-1}w} &= q^{ib^{-1}w} \mathbf{e}_2^{ib^{-1}w} \mathbf{e}_1 + q^{\frac{1}{2}}(q^{-ib^{-1}w} - q^{ib^{-1}w}) \mathbf{e}_2^{ib^{-1}w-1} \mathbf{e}_{12} \\ &= e^{-\pi b w} \mathbf{e}_2^{ib^{-1}w} \mathbf{e}_1 + q^{\frac{1}{2}}(e^{\pi b w} - e^{-\pi b w}) \mathbf{e}_2^{ib^{-1}w-1} \mathbf{e}_{12}.\end{aligned}$$

The first term has  $\mathbf{e}_1$  commute again with  $\mathbf{e}_{12}^{ib^{-1}v}$  and pick up the factor  $q^{-ib^{-1}v} = e^{\pi b v}$  before absorbing into  $\mathbf{e}_1^{ib^{-1}u}$ , producing the  $S_b$  factor same as above. Hence after shifting by  $\frac{\mathbf{i}Q}{2}$  it gives overall

$$e(-w + v)(e(u - 2p_u) + e(-u - 2p_u)).$$

The second term gives an opposite shifting  $e(2p_w)$  in the variable  $w$ , which cancels the factor in front:

$$q^{\frac{1}{2}}(e^{\pi b(w-ib)} - e^{-\pi b(w-ib)}) \frac{S_b(Q + \mathbf{i}w)}{S_b(Q + \mathbf{i}w + b)} = iq^{\frac{1}{2}}.$$

The shifting  $e(-2p_v)$  in  $v$  by  $\mathbf{e}_{12}$  then provides the same factor as before, as well as the extra number  $e^{-\frac{\pi Q(ib)}{2}} = (iq^{\frac{1}{2}})^{-1}$  from the auxiliary term  $e^{-\frac{\pi Qv}{2}}$  which cancels the above number. After shifting by  $\frac{iQ}{2}$  we then obtain

$$e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w).$$

Overall, the action of  $\mathbf{e}_1$  is then given by

$$\mathbf{e}_1 = e(-w + v + u - 2p_u) + e(-w + v - u - 2p_u) + e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w). \quad (4.11)$$

We will show that this is unitary equivalent to the positive representation  $\mathcal{P}$ .

Finally, the action of  $K_i$  is easy to read out. Multiplication from the left, each  $K_i$  commute with the root vectors  $\mathbf{e}_j$  and pick up a factor of the form  $(q^k)^{ib^{-1}u} = e(-ku)$ , before absorbing into the  $K$  terms. We find that the action is given by

$$\begin{aligned} K_1 &= e(-2u - v + w - 2p_{s_1} + \mathbf{i}Q), \\ K_2 &= e(u - v - 2w - 2p_{s_2} + \mathbf{i}Q). \end{aligned}$$

After shifting  $(u, v, w)$  by  $\frac{iQ}{2}$ , we have a positive action of  $K_i$ . Then by Proposition 4.4, we can ignore the  $p_{s_i}$  factors.

In summary, we obtained

$$\begin{aligned} \mathbf{e}_1 &= e(-w + v + u - 2p_u) + e(-w + v - u - 2p_u) + e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w), \\ \mathbf{e}_2 &= e(w - 2p_w) + e(-w - 2p_w), \\ K_1 &= e(-2u - v + w), \\ K_2 &= e(u - v - 2w). \end{aligned}$$

We can do the same procedure similarly for non-simply-laced case. For Type  $B_2$ , using  $w_0 = s_1 s_2 s_1 s_2$ , we define

$$\vec{F} = \int_C F(u, v, w) e^{-\pi Q_s v - \frac{\pi Q u}{2}} \frac{\mathbf{e}_2^{ib^{-1}w}}{S_b(Q + \mathbf{i}w)} \frac{\mathbf{e}_{12}^{ib_s^{-1}v}}{S_{b_s}(Q_s + \mathbf{i}v)} \frac{\mathbf{e}_X^{ib^{-1}u}}{S_b(Q + \mathbf{i}u)} \frac{\mathbf{e}_1^{ib_s^{-1}t}}{S_{b_s}(Q_s + \mathbf{i}t)} dt du dv dw, \quad (4.12)$$

where  $\mathbf{e}_X := T_1 T_2(\mathbf{e}_1)$  and  $b^2 = 2b_s^2$ . Then using successively Proposition 2.28, we

arrive at

$$\begin{aligned}
\mathbf{e}_1 &= e(-w + u + t - 2p_t) + e(-w + u - t - 2p_t) \\
&\quad + e(-w + v + u - 2p_u + 2p_v) + e(-w + v - u - 2p_u + 2p_v) \\
&\quad + e(v - 2p_v + 2p_w) + e(-v - 2p_v + 2p_w), \\
\mathbf{e}_2 &= e(w - 2p_w) + e(-w - 2p_w), \\
K_1 &= e(-2t - u + w), \\
K_2 &= e(2t - 2v - 2w).
\end{aligned}$$

From now on we will ignore the multiplicity module  $M_K$  (4.7) and only work on the  $L^2(\mathbb{R}^N)$  part.

#### 4.4 Twisting

We have seen from the last section that the action is positive self-adjoint, and each term is very closely related to the standard form  $[u]e(-2p_u)$  except for a small twist (cf. first and second term of (4.11)). In the simply-laced case this comes from the commutation relation arising from any relations that is equivalent (Lusztig transformed) to

$$\mathbf{e}_i \cdot \mathbf{e}_j^{b^{-1}}$$

and locally the action looks exactly like (4.11). Now using Lemma 2.20 successively, we transform the action by

$$\mathbf{e}_i \mapsto \Phi_1 \mathbf{e}_i \Phi_1^*,$$

$$\Phi_1 = g_b(e(w - u - 2p_u + 2p_v - 2p_w)g_b^*(e(-w - u - 2p_u + 2p_v - 2p_w)),$$

This absorb the second term of  $\mathbf{e}_1$  to the third term, and spit out the symmetric term from the fourth term, giving

$$\mathbf{e}_1 = [-w + v + u]e(-2p_u) + [v]e(-2p_v + 2p_w), \quad (4.13)$$

$$\mathbf{e}_2 = [w]e(-2p_w), \quad (4.14)$$

$$K_1 = e(-2u - v + w), \quad (4.15)$$

$$K_2 = e(u - v - 2w), \quad (4.16)$$

which is now symmetric in the quantum variables. On the other hand, one check that this action preserves the action of  $\mathbf{e}_2$ : it spits out an auxiliary term from the first term, and then reabsorb back to the second term.

For doubly-laced case, this transformation is given by

$$\begin{aligned}
\Phi_2 &= g_{b_s}(e(-t + v - 2p_t + 2p_u - 2p_v)g_{b_s}^*(e(-t - v - 2p_t + 2p_u - 2p_v) \\
&\quad \circ g_b(e(w - u - 2p_u + 4p_v - 2p_w))g_b^*(e(-w - u - 2p_u + 4p_v - 2p_w))),
\end{aligned}$$

which gives us

$$\mathbf{e}_1 = [t + u - w]e(-2p_t) + [u + v - w]e(-2p_u + 2p_v) + [v]e(-2p_v + 2p_w), \quad (4.17)$$

$$\mathbf{e}_2 = [w]e(-2p_w), \quad (4.18)$$

$$K_1 = e(-2t - u + w), \quad (4.19)$$

$$K_2 = e(2t - 2v - 2w). \quad (4.20)$$

In general, we can define transformations  $\Phi_k$  which preserves the action of all  $\mathbf{e}_i, i \neq k$ , and bringing the action of  $\mathbf{e}_k$  to the standard form (2.23). This transformation depends on the fact that there is a variable  $v_i$  of minimal index, corresponding to  $\mathbf{e}_{ij}$  in (2.66) which does not pick up any  $e^{\pi b x}$  factor, hence in the standard form (2.23). Then one successively apply the above transformation to remove the twisting within  $\mathbf{e}_k$  in the order where the commutation relation takes place between adjacent pair of root vectors (or their Lusztig transforms).

Finally, we note that under the change of variables

$$T : \begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} v - u \\ u \\ w \end{pmatrix}$$

with the corresponding action on  $p$  given by the transpose inverse:

$$\begin{pmatrix} p_u \\ p_v \\ p_w \end{pmatrix} \mapsto \begin{pmatrix} p_u + p_v \\ p_u \\ p_w \end{pmatrix}$$

the representation is exactly the same as the one given in Proposition 2.10. In particular

**Proposition 4.5.** *The GNS representation is independent of the choice of longest element  $w_0 \in W$ .*

*Proof.* For a change of words of  $w_0$ , it means the change in Lusztig's transformation  $\dots T_i T_j T_i \dots \longrightarrow \dots T_j T_i T_j \dots$  which locally is equivalent to the transformation given by the case of type  $A_2$ . Hence any such transformation is given by the unitary equivalence

$$T^* \circ \Phi \circ T,$$

where  $\Phi$  is the unitary equivalence from the positive representation [14], while  $T$  is the transformation matrix above. One can then calculate explicitly the map as

$$\Psi = T' \circ g_b(e(w - u + 2p_u - 2p_v + 2p_w)g_b^*(e(-w + u + 2p_u - 2p_v + 2p_w)), \quad (4.21)$$

where

$$T' : \begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} -u + v + w \\ u \\ v \end{pmatrix}$$



is a unitary transformation. Finally note that the transformation  $\Psi$  is essentially the inverse transpose of  $\Phi$  (interchanging the role of  $p_i$ 's and  $v_i$ 's), hence it is consistent with the choice of paths as established in [14]. Similarly we have a generalized transformation for the doubly-laced case given by ratios of 6 quantum dilogarithms followed by a linear transformation, which is again the inverse transpose of the transformation found in [15].  $\square$

With the procedure above, we can construct the GNS representation  $\mathcal{P}_{GNS}$  for  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$  of arbitrary type (using Proposition 2.28 etc.). In particular,  $\mathcal{P}_{GNS}$  is irreducible, and satisfies all properties required for a positive representation (positivity and transcendental relations (2.20)), hence this gives us another family of irreducible positive representations for  $\mathcal{U}_{q\tilde{q}}(\mathfrak{b}_{\mathbb{R}})$ . From the expression of  $\Psi$ , which is essentially the inverse transpose of  $\Phi$ , we have

**Theorem 4.6.** *The expressions of  $e_i$  obtained from the GNS construction (after twisting) is precisely the positive representations with all the terms  $[X]e(-2p_X)$  replaced by  $[Y]e(-2p_X)$  (cf. Definition 2.7). The action of  $K_i$  is simply obtained from a linear change of variables by the Lusztig's isomorphism corresponding to reflections by simple roots.*

Compare the expressions (4.13)-(4.20) with the expressions from Proposition 2.10 and 2.11.

## 5 Equivalence of representation $\mathcal{P}_{GNS} \simeq \mathcal{P}$

So far we have constructed a family of representations  $\mathcal{P}_{GNS}$  from the GNS-representation. In this section we will show that the representation is unitary equivalent to the one constructed earlier in [14, 15].

For type  $A_n$ , it turns out we can do this directly by a change of variables.

**Theorem 5.1.** *For type  $A_n$ , the GNS representation  $\mathcal{P}_{GNS}$  constructed by left multiplication on  $\mathcal{A} = \mathcal{U}_{q\tilde{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$  is unitary equivalent to the positive representation  $\mathcal{P}_{\lambda} \simeq \mathcal{P}$  of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to its Borel part.*

*Proof.* It suffices to consider a particular choice of the longest element. We fixed the longest element  $w_0 \in W$  to be the standard expression given in [14]:

$$w_0 = s_n \ s_{n-1} \dots s_1 \ s_n s_{n-1} \dots s_2 \ \dots \ s_n s_{n-1} s_n.$$

Then the GNS representation constructed in this section is unitary equivalent to the positive representation  $\mathcal{P}$  by a change of variables (using the notation from Definition 2.4)

$$\begin{aligned} u_k^m &\mapsto u_{n+1-m}^{n+1-k}, \\ u_k^m &\mapsto u_{n+1-m}^{n+1-k} - u_{n+2-m}^{n+2-k}, \quad m > 1. \end{aligned}$$

$\square$

For other types, we employ the following trick together with the Lusztig's isomorphisms  $T_i$ .

**Lemma 5.2.** *There exists a unitary transformation such that the action of  $\mathbf{e}_i$  is transformed into  $q_i^{-1}K_i\mathbf{e}_i$ .*

*Proof.* If we add  $\prod_{k=1}^N K_{\alpha_k}^{t_k}$  in front of  $\vec{F} \in \mathcal{A}$  in Definition 4.3, then the action of  $\mathbf{e}_i$  will pick up the correct factors according to  $K_i^{-1}$ . But if we commute these factors through the  $\mathbf{e}_{\alpha_k}$  to the right and absorb into the  $K$  terms, then on the representation space  $\mathcal{P}$  this is equivalent to multiplying by an exponential function of the form

$$e^{\pi i h(t_1, \dots, t_N)},$$

where  $h(t_1, \dots, t_N)$  is a homogeneous quadratic function in the variables  $t_k$ . In particular this is a unitary transformation.  $\square$

**Lemma 5.3.** *The Lusztig's isomorphism (cf. Theorem 2.24) corresponding to the longest word of the Weyl group  $w_0 = s_{i_1} \dots s_{i_N}$  given by  $T_{i_1} \dots T_{i_N}$  transform*

$$q_i^{-1}K_i\mathbf{e}_i \mapsto \mathbf{f}_{\sigma(i)},$$

where  $\mathbf{f}_i$  is the lower Borel generators,  $\sigma$  is the identity automorphism for  $\mathfrak{g}$  of type  $B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2$ , and the unique index 2 automorphisms on the simple roots for  $\mathfrak{g}$  of type  $A_n, D_{2n+1}$  and  $E_6$ .

*Proof.* It follows directly from the action given by Theorem 2.24, the properties from Proposition 2.25 and the fact that the longest Weyl element acts on simple roots as  $\alpha_i \mapsto -\alpha_{\sigma(i)}$ .  $\square$

Since we have seen from Theorem 4.6 that the action given by  $\mathcal{P}_{GNS}$  is precisely obtained by flipping  $[X]e(-2p_Y) \rightarrow [Y]e(-2p_X)$ , the Lusztig's isomorphism  $T_i$  modified according to this flip, will map the action of  $\mathbf{e}_i$  on  $\mathcal{P}_{GNS}$  to the corresponding action of  $\mathbf{f}_i$  given by Theorem 2.8. More precisely, we obtain

$$\mathbf{e}_i = \sum_{k=1}^n [-u_i^k] e \left( \sum_{j=1}^{v(i,k)-1} a_{i,r(j)} p_j + p_i^k \right), \quad (5.1)$$

with the corresponding roots and reduced expression of  $w_0$  relabeled. Recall that  $a_{ij}$  is the Cartan matrix,  $r(j)$  is the root label corresponding to the variable  $p_j$  (which is the shift in the variable  $v_j$ ), and  $v(i,k)$  is the index such that  $u_i^k = v_{v(i,k)}$  (i.e. we sum all the terms appearing to the left of  $p_i^k$ ) and we used the abuse of notation from Definition 2.7.

**Theorem 5.4.** *There exists a unitary transformation such that (5.1) is unitary equivalent to (2.25).*

*Proof.* Notice that for each term in the expression of  $\mathbf{f}_i$ , the variable  $p_i^k$  with the maximal index  $v(i, k)$  is unique, and these cover all variables  $(p_1, \dots, p_N)$ . Therefore we apply the transformation successively from the largest  $v(i, k)$  index by

$$b_i p_i^k \mapsto b_i p_i^k - \sum_{j=1}^{v(i,k)-1} a_{i,r(j)} b_j p_j, \quad (5.2)$$

which induces the corresponding change of variables

$$b_j v_j \mapsto b_j v_j + a_{i,r(j)} b_i v_i^k \quad (5.3)$$

for  $1 \leq j \leq v(i, k) - 1$ . It is clear that under this transformation, we reduce all exponential terms to  $e(2p_i^k)$ . On the other hand, it is easy to see that under (5.3) the quantum factors are changed to

$$[-u_i^k] \mapsto \left[ -u_i^k - \sum_{j=v(i,k)+1}^N a_{i,r(j)} v_j \right].$$

But this is precisely the action of  $\mathbf{f}_i$  corresponding to the longest element  $w_0$  with the reversed reduced expression

$$w_0 = s_{i_N} \dots s_{i_1}.$$

Since the positive representation is independent of the reduced expression of  $w_0$ , we have proved the unitary equivalence.  $\square$

Finally, since  $K_i$  depends only linearly on the variables (inside an exponential), it is uniquely determined by the shifts  $e(2p_i^k)$  appearing in the action of  $\mathbf{f}_i$ . In particular, under the Lusztig's transformation and the above unitary equivalence,  $K_i$  from  $\mathcal{P}_{GNS}$  transform precisely to  $K_{\sigma(i)}^{-1}$  as given in Theorem 2.8.

Therefore we can now state our Main Theorem.

**Theorem 5.5.** *For any simple Lie algebra  $\mathfrak{g}$ , the GNS representation  $\mathcal{P}_{GNS}$  constructed by left multiplication on  $\mathcal{A} = \mathcal{U}_{q\bar{q}}^{C^*}(\mathfrak{b}_{\mathbb{R}})$  is unitary equivalent to the positive representation  $\mathcal{P}_{\lambda} \simeq \mathcal{P}$  of  $\mathcal{U}_{q\bar{q}}(\mathfrak{g}_{\mathbb{R}})$  restricted to its Borel part.*

**Corollary 5.6.** *The tensor products of the positive representations  $\mathcal{P}$  restricted to the Borel part is unitary equivalent to the tensor product of itself with a trivial multiplicity module:*

$$W : \mathcal{P} \otimes \mathcal{P} \simeq M \otimes \mathcal{P}, \quad (5.4)$$

where the underlying Hilbert spaces are all  $\mathcal{P} \simeq M \simeq L^2(\mathbb{R}^N)$ ,  $N = l(w_0)$ .

*Proof.* Since we know that  $\mathcal{P} \simeq \mathcal{P}_{GNS}$  and the GNS-space  $\mathcal{H} \simeq \mathcal{P}_{GNS} \otimes M_K$ , this follows immediately from Proposition 2.31 of the property of the multiplicative unitary. The unitary transformation will be given by the multiplicative unitary  $W$  after factoring the trivial intertwiners on  $M_K$ , for example, by supplying the delta distributions on the corresponding variables.  $\square$

## 6 Example

In this section, we write explicitly the representations and decompositions for type  $A_1, A_2, A_3$ , which provide us a useful basic framework when we try to deal with the explicit construction of quantum higher Teichmüller theory.

### 6.1 Type $A_1$

Let us first consider the simplest case where  $\mathfrak{g}$  is of type  $A_1$ . The Borel part of  $\mathcal{U}_{q\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$  in this case is nothing but the quantum plane  $\mathcal{B}_{q\tilde{q}}$ , which is studied extensively in [11] and [13]. Nonetheless, it will be illustrative to see how the analysis work with respect to the theory of multiplicative unitary from Proposition 2.31.

First, we can do a transformation to bring the representation  $\mathcal{P}_{GNS}$  into a canonical form. By applying

$$e^{\frac{\pi i u^2}{2}} g_b(e(-2u)),$$

the action  $\mathbf{e} = [u]e(-2p_u)$  is transformed to  $\mathbf{e} = e(-2p_u)$ . Also take into account the positivity of  $K$  under the shift  $u \mapsto u + \frac{iQ}{2}$ , in terms of the  $C_*$ -algebraic Borel part  $\mathcal{A}$ , the expression is given by

$$\mathcal{A} \ni f = \int_C f(u, s) \mathbf{e}^{ib^{-1}u} K^{ib^{-1}s} e^{\pi Qs} duds,$$

Now recall from (2.76), the decomposition is given by the multiplicative unitary, defined by

$$W^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1)).$$

One can compute the action of  $W^*$  on  $f(u, s)g(v, t)$  to be:

$$\begin{aligned} W^*(f \otimes g) &= \int f(u - \tau, s - t + \tau) g(v + \tau, t - \tau) \frac{G_b(Q + \mathbf{i}v + \mathbf{i}\tau)e^{-\pi Q\tau}}{G_b(Q + \mathbf{i}v)G_b(Q + \mathbf{i}\tau)} e^{-2\pi \mathbf{i}(u-\tau)(t-\tau)} d\tau \\ &= g_b^*(e(-2p_u - 2p_t + 2p_v - v)) g_b^*(e(-2p_u - 2p_t + 2p_v)) \circ e^{-2\pi \mathbf{i}t(p_s + u)}, \end{aligned}$$

where we have shifted  $u, v$  by  $\frac{iQ}{2}$ , according to the Haar measure (4.4) to make the map unitary. Now recall we can transform such that the action of  $\mathcal{P}_{GNS}$  does not depend on  $s$  and  $t$ . This can be achieved by taking the Fourier transform on  $s$  and  $t$ , and shift the variables by  $u \mapsto u - s, v \mapsto v - t$ . We obtain finally

$$W^* = g_b^*(e(-2p_u + 2p_v - 2v)) e^{2\pi \mathbf{i}p_v u} g_b^*(e(-2p_u - 2t)) \circ e^{2\pi \mathbf{i}p_t u}. \quad (6.1)$$

Since the variables  $s, t$  correspond to the trivial module  $M_K$ , we can obtain the required unitary transformation on  $\mathcal{P}_{GNS}$  by supplying the delta distribution  $\delta(s)\delta(t)$ , such that upon acting by  $e^{2\pi \mathbf{i}p_t u}$ , we get

$$W^* = g_b^*(e(-2p_u + 2p_v - 2v)) e^{2\pi \mathbf{i}p_v u} g_b^*(e(-2p_u + 2u)).$$

Finally, acting  $W^*$  on  $1 \otimes \mathcal{B}_q$ , the term

$$g_b^*(e(-2p_u + 2u))$$

has no effects and acts as a trivial intertwiner. Hence factorizing this action, we simplify the unitary transformation for  $\mathcal{P}_{GNS}$ :

$$W^* = g_b^*(e(-2p_u + 2p_v - 2v))e^{2\pi i p_v u}, \quad (6.2)$$

$$\begin{aligned} 1 \otimes \mathbf{e} &\mapsto \Delta(\mathbf{e}) = \mathbf{e} \otimes K + 1 \otimes \mathbf{e}, \\ 1 \otimes K &\mapsto \Delta(K) = K \otimes K, \end{aligned}$$

which is precisely the integral transformation described in [11, 13].

## 6.2 Type $A_2$

As we have seen, in general it is hard to calculate the equivalence transformation from the multiplicative unitary. However, using the functional properties of the quantum dilogarithms, the required transformation can be explicitly calculated for Type  $A_n$ . We illustrate the situation for small cases. Let us consider the next simplest case  $\mathfrak{g} = \mathfrak{sl}_3$ . First we simplify the expression of  $\mathcal{P}_\lambda \simeq \mathcal{P}$  from Proposition 2.10, according to Remark 3.3.

**Lemma 6.1.**  *$\mathcal{P}$  is unitary equivalent to the following expression:*

$$\begin{aligned} e_1 &= e(v - w - 2p_v) + e(u - 2p_u - 2p_v + 2p_w), \\ e_2 &= e(w - 2p_w), \\ K_1 &= e(u - 2v + w), \\ K_2 &= e(-2u + v - 2w). \end{aligned}$$

*Proof.* Apply  $\Phi_1$  (i.e.  $\mathbf{e}_i \mapsto \Phi_1 \mathbf{e}_i \Phi_1^*$ ), where

$$\Phi_1 = g_b(e(-u - v + w + 2p_u - 2p_w))g_b(e(u - v - w - 2p_u + 2p_w))g_b(e(-2u))g_b(e(-2w)). \quad (6.3)$$

□

Using this expression, now let us consider the tensor product representation  $\mathcal{P} \otimes \mathcal{P}$ :

$$\begin{aligned} \Delta(\mathbf{e}_1) &= e(v - w + u' - 2v' + w' - 2p_v) + e(u + u' - 2v' + w' - 2p_u - 2p_v + 2p_w) + \\ &\quad e(v' - w' - 2p'_v) + e(u' - 2p'_u - 2p'_v + 2p'_w), \\ \Delta(\mathbf{e}_2) &= e(w - 2u' + v' - 2w' - 2p_w) + e(w' - 2p'_w), \\ \Delta(K_1) &= e(u - 2v + w + u' - 2v' + w'), \\ \Delta(K_2) &= e(-2u + v - 2w - 2u' + v' - 2w'). \end{aligned}$$

where the  $'$  variables denote the 2nd component.

**Lemma 6.2.** *It is unitary equivalent to:*

$$\begin{aligned}\Delta(\mathbf{e}_1) &\simeq 1 \otimes \mathbf{e}_1, \\ \Delta(\mathbf{e}_2) &\simeq 1 \otimes \mathbf{e}_2, \\ \Delta(K_1) &\simeq K_1 \otimes K_1, \\ \Delta(K_2) &\simeq K_2 \otimes K_2.\end{aligned}$$

*Proof.* Apply  $\Phi_2$  where

$$\begin{aligned}\Phi_2 = & g_b(e(w - u' - 2w' - 2p_w - 2p'_u + 4p'_w) \circ \\ & g_b(e(u - 2v' + w' - 2p_u - 2p_v + 2p_w + 2p'_u + 2p'_v - 2p'_w) \circ \\ & g_b(e(v - w - 2v' + w' - 2p_v + 2p'_u + 2p'_v - 2p'_w) \circ \\ & g_b(e(w - 2u' + v' - 3w' - 2p_w + 2p'_w))).\end{aligned}$$

□

**Corollary 6.3.** *From the explicit expression of Lemma 6.1 and the fact that the positive representation is irreducible, the transformation  $\Phi_2$  can be written in a way independent of the representation:*

$$\Phi_2 = g_b(q^2 \mathbf{e}_2 \otimes K_2 \mathbf{e}_{21} \mathbf{e}_{12}^{-1} \mathbf{e}_2^{-1}) g_b(q^2 \mathbf{e}_{21} \mathbf{e}_2^{-1} \otimes K_1 \mathbf{e}_{21}^{-1} \mathbf{e}_2) g_b(q \mathbf{e}_{12} \mathbf{e}_2^{-1} \otimes K_1 \mathbf{e}_{21}^{-1} \mathbf{e}_2) g_b(q \mathbf{e}_2 \otimes K_2 \mathbf{e}_2^{-1}).$$

Hence this transformation gives directly the intertwiners

$$\begin{aligned}1 \otimes \mathbf{e}_i &\simeq \Delta(\mathbf{e}_i), \\ K \otimes K &\simeq K \otimes K.\end{aligned}$$

Here we put the appropriate  $q$  factors to make the arguments positive. Note that the inverse operator is defined in the natural way. Since it is positive and all the terms  $q$ -commute, the expression of the transformation is still well-defined.

Finally, note that  $u, v, w$  from first component of  $K_i$  now acts as constant, so we can treat them as  $\lambda$ 's and use Theorem 3.1 to remove them. We obtain at the end

$$\begin{aligned}\Delta(\mathbf{e}_1) &\simeq 1 \otimes \mathbf{e}_1 = e(v' - w' - 2p'_v) + e(u' - 2p'_u - 2p'_v + 2p'_w), \\ \Delta(\mathbf{e}_2) &\simeq 1 \otimes \mathbf{e}_2 = e(w' - 2p'_w), \\ \Delta(K_1) &\simeq 1 \otimes K_1 = e(u' - 2v' + w'), \\ \Delta(K_2) &\simeq 1 \otimes K_1 = e(-2u' + v' - 2w').\end{aligned}$$

In other words, when restricting to the Borel part, we have

$$\mathcal{P}_{\lambda_1} \otimes \mathcal{P}_{\lambda_2} \simeq \mathcal{P} \otimes \mathcal{P} = L^2(\mathbb{R}^3) \otimes \mathcal{P}.$$

### 6.3 Type $A_3$

Let the variables be  $\{r, s, t, u, v, w\}$  and we use  $w_0 = s_3 s_2 s_1 s_3 s_2 s_3$ . Again we have the simple form by removing the quantum brackets from Remark 3.3.

**Lemma 6.4.**  $\mathcal{P}$  is unitary equivalent to the following expression:

$$\begin{aligned} \mathbf{e}_1 &= e(t - v - p_t) + e(s - u - p_s - p_t + p_v) + e(r - p_r - p_s - p_t + p_u + p_v), \\ \mathbf{e}_2 &= e(v - w - p_v) + e(u - p_u - p_v + p_w), \\ \mathbf{e}_3 &= e(w - p_w), \\ K_1 &= e(s - 2t + v), \\ K_2 &= e(r - 2s + t + u - 2v + w), \\ K_3 &= e(-2r + s - 2u + v - 2w). \end{aligned}$$

*Proof.* Apply the transformation to  $\mathcal{P}$ :

$$\begin{aligned} &g_b(e(s - t - w - 2p_s - 2p_u + 2p_v + 2p_w))g_b(e(r - s - v + w - 2p_r + 4p_u - 2p_w), \\ &\circ g_b(e(-r - t + v + 2p_r + 2p_s - 2p_u - 2p_v))g_b(e(-r - s + u + 2p_r - 2p_u), \\ &\circ g_b(e(r - s - u - 2p_r + 2p_u))g_b(e(-2r)) \circ \Phi_1, \end{aligned}$$

where  $\Phi_1$  is given by (6.3) for the rank 2 case.  $\square$

**Lemma 6.5.** It is unitary equivalent to an expression where  $\mathbf{e}_i$  does not depend on the first component:

$$\begin{aligned} \mathbf{e}_1 &= e(t' - v' - p'_t) + e(s' - u' - p'_s - p'_t + p'_v) + e(r' - p'_r - p'_s - p'_t + p'_u + p'_v), \\ \mathbf{e}_2 &= e(v' - w' - p'_v) + e(u' - p'_u - p'_v + p'_w), \\ \mathbf{e}_3 &= e(w' - p'_w), \\ K_1 &= e(s - 2t + v + s' - 2t' + v'), \\ K_2 &= e(r - 2s + t + u - 2v + w + r' - 2s' + t' + u' - 2v' + w'), \\ K_3 &= e(-2r + s - 2u + v - 2w - 2r' + s' - 2u' + v' - 2w'). \end{aligned}$$

*Proof.* Apply the following transformation

$$\begin{aligned} &g_b(e(r - r' + s' + v' - 2t' - 2p_r - 2p_s - 2p_t + 2p_u + 2p_v + 2p'_r + 2p'_s + 2p'_t - 2p'_u - 2p'_v)) \\ &\circ g_b(e(u + 2r' - 3s' + u' - 2v' + w' + t' - 2p_u - 2p_v + 2p_w - 2p'_r + 4p'_u + 2p'_v - 2p'_w)) \\ &\circ g_b(e(w - 2r' + 2s' - t' - u' + v' - 2w' - 2p_w - 2p'_s - 2p'_u + 2p'_v + 4p'_w)) \\ &\circ g_b(e(v - w + 2r' - 3s' + t' + u' - 2v' + w' - 2p_v - p'_r + 4p'_u + 2p'_v - 2p'_w)) \\ &\circ g_b(e(s - r' + s' - 2t' + v' - 2p_s - 2p_t + 2p_v + 2p'_r + 2p'_s + 2p'_t - 2p'_u - 2p'_v)) \\ &\circ g_b(e(t - v - r' + s' - 2t' + v' - 2p_t + 2p'_r + 2p'_s + 2p'_t - 2p'_u - 2p'_v)) \\ &\circ g_b(e(w - 2r' + s' - u' - 2w' - 2p_w - 2p'_u + 4p'_w)) \\ &\circ g_b(e(u + r' - 2s' + t' - 2v' + w' - 2p_u - 2p_v + 2p_w + 2p'_u + 2p'_v - 2p'_w)) \\ &\circ g_b(e(v - w + r' - 2s' + t' - 2v' + w' - 2p_v + 2p'_u + 2p'_v - 2p'_w)) \\ &\circ g_b(e(w - 2r' + s' - 2u' + v' - 3w' - 2p_w + 2p'_w)). \end{aligned}$$

Compare with type  $A_2$ , note that the last 4 terms are precisely the same if we ignore the additional variables  $r, s, t$  in this higher rank case, since we are eliminating terms inductively from the  $A_2$  expression of  $w_0 = \dots s_3 s_2 s_3$ . Furthermore, as in Corollary 6.3, we can rewrite the transformation up to this point using only the root vectors which can be expressed as single exponentials, namely  $\{\mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{32}, \mathbf{e}_{123}, \mathbf{e}_{321}, K_1, K_2, K_3\}$ , making it representation independent.  $\square$

Finally, again applying Theorem 3.1 to remove the variables on  $K_i$  corresponding to the first component, we obtain a decomposition

$$\mathcal{P}_{\lambda_1} \otimes \mathcal{P}_{\lambda_2} \simeq \mathcal{P} \otimes \mathcal{P} = L^2(\mathbb{R}^6) \otimes \mathcal{P}.$$

**Remark 6.6.** For type  $A_n$ , there is a pattern of this series of transformations computed inductively by *Mathematica*, and it seems to be related to the Heisenberg double [19]. However, for other types in general we did not establish any explicit construction of such transformations.

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